NOTICE: When government or other drawings, specifications or other data are used for any purpose other than in connection with a definitely related government procurement operation, the U. S. Government thereby incurs no responsibility, nor any obligation whatsoever; and the fact that the Government may have formulated, furnished, or in any way supplied the said drawings, specifications, or other data is not to be regarded by implication or otherwise as in any manner licensing the holder or any other person or corporation, or conveying any rights or permission to manufacture, use or sell any patented invention that may in any way be related thereto.

63-1-4

AFCRL-62-504

NEW YORK UNIVERSITY
Courant Institute of Mathematical Sciences
Division of Electromagnetic Research

RESEARCH REPORT No. EM 174

The Three-dimensional Inverse Scattering Problem

IRVIN KAY

Contract No. AF 19(604)3495
Project 5631
Task 563104
JUNE, 1962

Best
Available
Copy

New York University
Courant Institute of Mathematical Sciences
Division of Electromagnetic Research

Research Report No. EM-174

The Three-Dimensional Inverse Scattering Problem

Irvin Kay

July, 1962

Contract No. AF 19(604)-3495
Project 5631
Task 563104

Irvin Kay
Irvin Kay

Morris Kline
Morris Kline, Director

Prepared for:
Electronics Research Directorate
Air Force Cambridge Research Laboratories
Office of Aerospace Research
USAF
Bedford, Massachusetts

Requests for additional copies by Agencies of the Department of Defense, their contractors, and other Government agencies should be directed to the:

ARMED SERVICES TECHNICAL INFORMATION AGENCY
DOCUMENTS SERVICE CENTER
ARLINGTON HALL STATION
ARLINGTON 12, VIRGINIA

Department of Defense contractors must be established for ASTIA services or have their 'need-to-know' certified by the cognizant military agency of their project or contract.
All other persons and organizations should apply to the:

U.S. DEPARTMENT OF COMMERCE
OFFICE OF TECHNICAL SERVICES
WASHINGTON 25, D.C.

1. Introduction

The purpose of this paper is to put together in a concrete form the mathematical theory of an inverse scattering problem which was originally formulated in quantum mechanical terms but which is related to the problem of determining the variation of electron density in an ionized gas such as the earth's ionosphere by means of radio sounding experiments. The theory described here was developed by H. E. Moses and the present writer and was originally presented rather generally and abstractly.[1,2] In the present article the definitions and proofs of theorems have been revised in an attempt to make the analysis understandable to a reader who may not be familiar with the terminology of quantum mechanics or of the associated Hilbert space theory. While the usual terminology employed by physicists has not been discarded completely here, definitions and relations are given in concrete terms whenever possible, and abstract relations are used for convenience of notation rather than for the purpose of obtaining great generality.

The physical problem at which the present work is aimed is to find the variation of electron density in a weakly ionized gas from a knowledge of the scattering amplitude resulting from the incidence of a plane electromagnetic wave. It is assumed that the relevant properties of the medium vary slowly enough so that a scalar wave describes the scattering phenomena with sufficient accuracy. It is also assumed that losses, ordinarily attributed to the collisions of the electrons with

heavy particles may be neglected. The effect of an external magnetic field which would cause anisotropy is neglected as well, so that the scattering is assumed to be isotropic as well as lossless.

A steady-state scalar field component $u(x,y,z)$ under these conditions will satisfy a differential equation

$$\Delta u + \left\{ k^2 - V(x,y,z) \right\} u = 0, \qquad \text{(A)}$$

having the same form as the time-independent Schrödinger equation of quantum mechanics.[3] The quantity k is the wave number and the function $V(x,y,z)$ depends upon the variation of electron density in the medium; in fact, $V(x,y,z)$ is the plasma frequency divided by the square of the velocity of light in vacuum.

The scattering problem associated with such a differential equation would be to calculate the scattering amplitude $T(x,y,z,\underline{k})$ defined by the asymptotic form of a solution u for which the incident wave is plane and uniform and propagates in the direction of the vector $\underline{k}$. That is, if

$$u_{inc.} = e^{i\underline{k}\cdot\underline{x}},$$

where $\underline{x}$ is the radius vector whose components are x,y,z, and at large distances

$$u \sim e^{i\underline{k}\cdot\underline{x}} + T(\underline{x},\underline{k}) \frac{e^{ikx}}{x}, \qquad \text{(B)}$$

where k and x are the magnitudes of the vectors $\underline{k}$ and $\underline{x}$, the inverse scattering problem would be to determine the function $V(x,y,z)$ in (A), given the scattering amplitude $T(\underline{x},\underline{k})$, or some part of it.

At the present time the inverse scattering problem for the system described precisely by equation (A), in three dimensions, is still open. In this paper we shall consider, instead of (A), the more general integro-differential equation

$$\text{(C)} \qquad \Delta u + k^2 u - \int v(\underline{x},\underline{x}')u(\underline{x}',\underline{k})d\underline{x}' = 0,$$

where the integration as indicated by the differential $d\underline{x}'$ is over all three-dimensional $\underline{x}'$ space. We assume that the integral operator whose kernel is $v(\underline{x},\underline{x}')$ is such that the asymptotic relation (B) still holds for some solution of (C). Physically, the differential equation (C) would govern electromagnetic wave propagation in a somewhat artificial plasma of a more general type than that characterized by (A). However, we allow the kernel $v(\underline{x},\underline{x}')$ to be a distribution, and in particular it could have the form

$$\text{(D)} \qquad v(\underline{x},\underline{x}') = V(\underline{x})\ \delta(\underline{x}-\underline{x}'),$$

whereupon the integro-differential equation (C) would be indentical with the differential equation (A).

From the point of view of the inverse scattering problem we would have to discover a method of solution which would automatically impose the condition (D) if we wished to solve the original problem

associated with (A). Instead, what we actually do here is to impose conditions that guarantee a unique solution of the inverse problem associated with (C) but which do not guarantee the form (D) in general, although the possibility is not excluded that for some given scattering operators the solution $v(\underline{x},\underline{x}')$ will have the form (D) and hence will apply to the problem associated with (A).

In section 2 of this article various conventions which are used throughout are listed for convenience and are referred to specifically later on whenever it seems appropriate to do so.

In section 3 an integral transform theorem analogous to the Fourier integral theorem and involving arbitrary solutions $u(\underline{x},\underline{k})$ of (C) is presented. This theorem requires the use of a certain weight function $w(\underline{k},\underline{k}')$ needed to normalize the particular function $u(\underline{x},\underline{k})$ which appears in each instance of the theorem. We calculate $w(\underline{k},\underline{k}')$ for the case in which $u(\underline{x},\underline{k})$ is the function satisfying the asymptotic condition (B). The discussion should be of some interest by itself for other more usual applications such as the formation of wave packets in the medium determined by (C) (or, in particular, by (A)) associated with normalized scattered plane waves.

Section (4) deals with an inverse scattering problem for (C) wherein conditions sufficient to guarantee uniqueness of the solution are imposed. It is hoped that the same or a similar formal setup can be used in the future to attack the inverse scattering problem associated directly with (A).

2. Notation

In this section we list a set of conventions for easy reference later on. The notation described here will be used consistently throughout the rest of this article.

a) An underlined letter such as $\underline{x}$ will represent a three-dimensional vector (x_1, x_2, x_3). The same letter x, not underlined, will represent the magnitude $\sqrt{x_1^2 + x_2^2 + x_3^2}$ of that vector.

b) Unless there are explicit limits, an integral will be over the entire domain of the variable occurring in the differential. Thus

$$\int f(x)dx \quad \text{means} \quad \int_0^\infty f(x)dx, \text{ while } \int f(\underline{x})d\underline{x} \text{ means } \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x_1, x_2, x_3)dx_1 dx_2 dx_3.$$

Multiple integrals will be designated by the nature of the differentials which occur in them, and a single integral sign will always be used. For example, $\int f(\underline{x},\underline{y})d\underline{x}d\underline{y}$ is a six-dimensional integral over the entire domain of the vectors $\underline{x}$ and $\underline{y}$.

c) We shall be concerned with linear operators in the form of integral operators over a suitable space of functions. The kernel of such an operator will be denoted by a lower case letter. When we wish to speak of the operator in the abstract, the corresponding capital letter will be used. Thus, Uf means the abstract operator U acting on the abstract linear element f, and the concrete representer of U will be $\int u(\underline{x},\underline{x}')f(\underline{x}')d\underline{x}'$.

d) The kernels allowed for integral operators can be distributions as well as smooth functions. Thus, in particular, the identity operator I is represented by a kernel which is a delta function:

$$If \equiv \int \delta(\underline{x} - \underline{x}')f(\underline{x}')d\underline{x}' = f(\underline{x}) \equiv f.$$

The Laplacian operator Δ can be represented by an integral operator whose kernel is given by $\Delta\delta(\underline{x} - \underline{x}') = \Delta'\delta(\underline{x} - \underline{x}')$, wherein the primed differential operator is taken with respect to the primed variables, and the unprimed operator with respect to the unprimed variables. For example, we have

$$\Delta f \equiv \Delta f(\underline{x}) = \int \left\{\Delta\delta(\underline{x} - \underline{x}')\right\} f(\underline{x}')d\underline{x}'.$$

e) We shall use the notation U^* for the Hermitian adjoint of U, but $u^*(\underline{x},\underline{x}')$ will mean the complex conjugate of the function $u(\underline{x},\underline{x}')$. Hence the following are equivalent:

$$U^*f \equiv \int u^*(\underline{x}',\underline{x})f(\underline{x}')d\underline{x}' .$$

f) Instead of functions of $\underline{x}$ the Fourier transforms of the functions will sometimes be used to represent the same linear space. The new functions, the transforms, will depend on vectors $\underline{k}$ instead of $\underline{x}$ and will be indicated by a circumflex. Thus

$$f \equiv f(\underline{x}), \quad f \equiv \hat{f}(\underline{k}) = \frac{1}{(2\pi)^{3/2}} \int e^{i\underline{k}\cdot\underline{x}} f(\underline{x})d\underline{x}.$$

In the case of an integral operator kernel we shall also write

$$\hat{u}(\underline{x},\underline{k}) = \frac{1}{(2\pi)^{3/2}} \int u(\underline{x},\underline{x}') \, e^{-i\underline{k}\cdot\underline{x}'} d\underline{x}'.$$

Because of the identity

$$\frac{1}{(2\pi)^3} \int e^{i\underline{k}\cdot(\underline{x}-\underline{x}')} d\underline{k} = \delta(\underline{x}-\underline{x}')$$

we have

$$\int u(\underline{x},\underline{x}')f(\underline{x}')d\underline{x}' = \int \hat{u}(\underline{x},\underline{k}) \, \hat{f}(\underline{k})d\underline{k}.$$

g) The Laplacian operator Δ which acts on the functions of $f(\underline{x})$ is represented in $\underline{k}$-space, the space of the Fourier transforms $\hat{f}(\underline{k})$, by the multiplicative operator $-k^2$ (cf. 2.a), that is, by the integral operator whose kernel is $-k^2\delta(\underline{k}-\underline{k}')$. Thus,

$$\Delta f(\underline{x}) = -\frac{1}{(2\pi)^{3/2}}\int k^2 e^{-i\underline{k}\cdot\underline{x}} \, \hat{f}(\underline{k})d\hat{\underline{k}} = -\frac{1}{(2\pi)^{3/2}}\int k^2\delta(\underline{k}-\underline{k}')e^{-i\underline{k}\cdot\underline{x}} \, \hat{f}(\underline{k}')d\underline{k}'d\underline{k}.$$

Hence we can write

$$\Delta \equiv -k^2 \equiv -k^2\delta(\underline{k}-\underline{k}')$$

in $\underline{k}$-space. Because of the relation

$$f(\underline{x})\delta(\underline{x}) = f(0)\delta(\underline{x})$$

any operator whose kernel has the form

$$\Gamma \equiv \gamma(\underline{k} - \underline{k}')\, \delta(k^2 - k'^2)$$

commutes with the Laplacian Δ. That is,

$$\Delta\Gamma = \Gamma\Delta,$$

for we have

$$-k^2\gamma(\underline{k} - \underline{k}')\delta(k^2 - k'^2) = -k'^2\gamma(\underline{k} - \underline{k}')\delta(k^2 - k'^2).$$

3. Eigenfunctions

Solutions $u_o(\underline{x},\underline{k})$ of the differential equation

(1) $$-\Delta\, u_o(\underline{x},\underline{k}) = k^2 u_o(\underline{x},\underline{k})$$

will be called eigenfunctions of the operator $-\Delta$. Each solution $u_o(\underline{x},\underline{k})$ will be regarded as the kernel of an integral operator which is a representer of some abstract operator U_o. This integral operator acts on the space of functions $\hat{f}(k)$ which are transformed by it into functions $f(\underline{x})$ belonging to the $\underline{x}$-representation. Thus, we shall have relations of the type

$$\varphi(\underline{x}) = \int u_o(\underline{x},\underline{k})\, \hat{f}(\underline{k})d\underline{k}.$$

A special case of this is the Fourier transform for which

$$u_o(\underline{x},\underline{k}) = \frac{1}{(2\pi)^{3/2}}\, e^{-i\underline{k}\cdot\underline{x}}$$

(cf. 2.f).

Similarly, we shall be concerned with operators U and corresponding eigenfunctions $u(\underline{x},\underline{k})$ which are solutions of the integro-differential equation

$$(2) \qquad -\Delta u(\underline{x},\underline{k}) + \int v(\underline{x},\underline{x}')u(\underline{x}',\underline{k})d\underline{x}' = k^2 u(\underline{x},\underline{k}).$$

In operator form (2) is

$$-\Delta U + VU = k^2 U.$$

Here again $u(\underline{x},\underline{k})$ is regarded as the kernel of an integral operator which transforms a function of $\underline{k}$ into a function of $\underline{x}$. In (2) the operator V whose kernel is $v(\underline{x},\underline{x}')$ may in special cases be a multiplicative operator; i.e., $v(\underline{x},\underline{x}')$ may have the form $\varphi(x)\delta(\underline{x}-\underline{x}')$ (cf. 2.d). Then (2) will have the usual form of the Schrödinger equation.

It is well known that the operator $-\Delta$ is self-adjoint and has a continuous spectrum consisting of zero and all positive real numbers when acting on a space, e.g., of square integrable functions $f(\underline{x})$. We shall assume that this is true of $-\Delta$ and in addition of $-\Delta + V$ without worrying about particular conditions on the "perturbation" V necessary to guarantee this property or, for that matter, about the precise nature of the space of functions $f(\underline{x})$.

We have given an operator interpretation of the eigenfunctions of (1) and (2) as transformations from $\underline{k}$ space to $\underline{x}$ space. The more concrete interpretation is also useful, especially in applications, in which we regard them as providing integral transform theorems analogous

to the Fourier integral theorem. We can then use them, for example, to describe the transient response* to the system described by (1) or (2). Then, the Fourier integral theorem provides

$$(3)\qquad f(\underline{x}) = \frac{1}{(2\pi)^{3/2}} \int e^{-i\underline{k}\cdot\underline{x}}\, \hat{f}(\underline{k})d\underline{k},$$

$$(4)\qquad \hat{f}(k) = \frac{1}{(2\pi)^{3/2}} \int e^{i\underline{k}\cdot\underline{x}}\, f(\underline{x})d\underline{x}.$$

If we set

$$U_o \equiv \frac{1}{(2\pi)^{3/2}} e^{-i\underline{k}\cdot\underline{x}},$$

then (3) and (4) imply the operator relation

$$(5)\qquad U_o U_o^* = U_o^* U_o = I,$$

where I is the identity. The relations (5) mean that U_o is unitary.

Similarly if we define the particular eigenfunction $u_+(\underline{x},\underline{k})$ of (2) as the unique† solution of the integral equation

$$(6)\quad u_+(\underline{x},\underline{k}) = \frac{1}{(2\pi)^{3/2}} e^{i\underline{k}\cdot\underline{x}} - \frac{1}{4\pi} \int \frac{e^{ik|\underline{x}-\underline{x}'|}}{|\underline{x}-\underline{x}'|} v(\underline{x}',\underline{x}'')u_+(\underline{x}'',\underline{k})d\underline{x}'d\underline{x}'',$$

then we can prove an analogous transform theorem in terms of $u_+(\underline{x},\underline{k})$

---

*The transient response would be given by multiplying a function $f(\underline{k})$ by $e^{i\omega t}u(\underline{x},\underline{k})$, where $\omega = ck$, and integrating over ω.

†The solution of the integral equation (6) is unique because of our assumptions about the operator $-\Delta + V$.

and $u_+^*(\underline{x},\underline{k})$. We can prove:

if

$$f(\underline{x}) = \int u_+(\underline{x},\underline{k})\varphi(\underline{k})d\underline{k}, \tag{7}$$

then

$$\varphi(\underline{k}) = \int u_+^*(\underline{x},k)f(\underline{x})d\underline{x}. \tag{8}$$

Relations (7) and (8) imply the relation

$$U_+U_+^* = U_+^*U_+ = I \tag{9}$$

for the operator U_+ corresponding to $u_+(\underline{x},\underline{k})$. Thus, U_+ is unitary.

Similar results hold for the eigenfunction $u_-(\underline{x},\underline{k})$ and the corresponding operator U_-, where $u_-(\underline{x},\underline{k})$ is defined as the solution of the integral equation

$$u_-(\underline{x},\underline{k}) = \frac{1}{(2\pi)^{3/2}} e^{i\underline{k}\cdot\underline{x}} - \frac{1}{4\pi}\int \frac{e^{-ik|\underline{x}-\underline{x}'|}}{|\underline{x}-\underline{x}'|} v(\underline{x}',\underline{x}'')u_-(\underline{x}'',\underline{k})d\underline{x}'d\underline{x}''. \tag{10}$$

We shall prove the assertion we have made concerning the eigenfunctions $u_\pm(\underline{x},\underline{k})$ by showing that, for example,

$$\int u_+^*(\underline{x},\underline{k})u_+(\underline{x},\underline{k}')d\underline{x} = \delta(\underline{k}-\underline{k}'). \tag{11}$$

Since we have assumed that the spectra of $-\Delta$ and $-\Delta + V$ are the same the self-adjointness of these operators will imply that U_+ has a right

inverse as well as a left one. Thus, because of (11) we shall have

$$U_+^* U_+ = U_+ U_+^* = I.$$

The arguments required to prove a similar theorem in terms of U_- are completely analogous, and we shall have finally

$$(12) \qquad U_\pm^* U_\pm = U_\pm U_\pm^* = I,$$

or

$$(13) \qquad \begin{aligned} \int u_\pm^*(\underline{x},\underline{k}) u_\pm(\underline{x},\underline{k}') d\underline{x} &= \delta(\underline{k}-\underline{k}'), \\ \int u_\pm(\underline{x},\underline{k}) u_\pm^*(\underline{x}',\underline{k}) d\underline{k} &= \delta(\underline{x}-\underline{x}'). \end{aligned}$$

Proof of (11): Before beginning the argument it will be useful to define

$$\psi(\underline{x},\underline{k}) = \int v(\underline{x},\underline{x}') u_+(\underline{x}',\underline{k}) d\underline{x}'.$$

We can now write in place of (6)

$$(14) \quad u_+(\underline{x},\underline{k}) = \frac{1}{(2\pi)^{3/2}} e^{i\underline{k}\cdot\underline{x}} - \frac{1}{4\pi} \int \frac{e^{ik|\underline{x}-\underline{x}'|}}{|\underline{x}-\underline{x}'|} \psi_+(\underline{x}',\underline{k}) dx'.$$

We shall also need the well-known identity

$$(15) \qquad -\frac{1}{4\pi} \frac{e^{\pm ik|\underline{x}-\underline{x}'|}}{|\underline{x}-\underline{x}'|} = \lim_{\epsilon\to 0} \frac{1}{(2\pi)^3} \int \frac{e^{i\underline{p}\cdot(\underline{x}-\underline{x}')}}{k^2 - p^2 \pm i\epsilon} d\underline{p},$$

which is proved in appendix I. The relation (15) has the meaning of a distribution; i.e., the limit is to be taken after an integration over the variable $\underline{x}'$:

We shall also require the identity

$$(16)\qquad \frac{1}{(a \pm i\epsilon)(b \pm i\epsilon)} = \frac{1}{(a+b) \pm 2i\epsilon} \left\{ \frac{1}{a \pm i\epsilon} + \frac{1}{b \pm i\epsilon} \right\},$$

which follows from a simple straightforward algebraic manipulation.

In addition we shall need

$$(17)\qquad -\frac{1}{4\pi} \int \frac{e^{-i\underline{k}\cdot\underline{x} + ik'|\underline{x}-\underline{x}'|}}{|\underline{x}-\underline{x}'|} (\underline{x}',\underline{k}) d\underline{x}' d\underline{x} = \lim_{\epsilon \to 0} \int \frac{e^{-i\underline{k}\cdot\underline{x}'}}{k'^2 - k^2 + i\epsilon} \psi(\underline{x}',\underline{k}') d\underline{x}'$$

and

$$(18)\qquad -\frac{1}{4\pi} \int \frac{e^{i\underline{k}'\cdot\underline{x} - ik|\underline{x}-\underline{x}'|}}{|\underline{x}-\underline{x}'|} \psi^*(\underline{x}',\underline{k}) d\underline{x}' d\underline{x} = \lim_{\epsilon \to 0} \int \frac{e^{i\underline{k}'\cdot\underline{x}'}}{k^2 - k'^2 - i\epsilon} \psi^*(\underline{x}',\underline{k}) d\underline{x}'.$$

Relations (17) and (18) follow easily from (15). Thus, for example

$$-\frac{1}{4\pi} \int e^{-i\underline{k}\cdot\underline{x} + ik'|\underline{x}-\underline{x}'|} \psi(\underline{x}',\underline{k}') d\underline{x}' d\underline{x}$$

$$= \lim_{\epsilon \to 0} \frac{1}{(2\pi)^3} \int \frac{e^{i\underline{p}\cdot(\underline{x}-\underline{x}') - i\underline{k}\cdot\underline{x}}}{k'^2 - k^2 + i\epsilon} \psi(\underline{x}',\underline{k}') d\underline{x}' d\underline{x}\, d\underline{p}$$

$$= \lim_{\epsilon \to 0} \int \frac{e^{-i\underline{p}\cdot\underline{x}'}}{k'^2 - p^2 + i\epsilon} \delta(\underline{p} - \underline{k}) \psi(\underline{x}',\underline{k}') d\underline{x}' d\underline{p}$$

$$= \lim_{\epsilon \to 0} \int \frac{e^{-i\underline{k}\cdot\underline{x}'}}{k'^2 - k^2 + i\epsilon} \psi(\underline{x}',\underline{k}') d\underline{x}'.$$

Finally, we shall require the identity

$$(19)\qquad \int u^*(\underline{y},\underline{k})\psi_+(\underline{y},\underline{k}')d\underline{y} = \int u(\underline{y},\underline{k}')\psi_+^*(\underline{y},\underline{k})d\underline{y}.$$

This relation follows from our assumption that the operator V is Hermitian, i.e.,

$$v(\underline{x},\underline{y}) = v^*(\underline{y},\underline{x}).$$

We have, in fact,

$$\int u^*(\underline{y},\underline{k})\psi(\underline{y},\underline{k}')d\underline{y} = \int u^*(\underline{y},\underline{k})v(\underline{y},\underline{x})u(\underline{x},\underline{k}')d\underline{y}d\underline{x}$$

$$= \int u(\underline{x},\underline{k}')v^*(\underline{x},\underline{y})u^*(\underline{y},\underline{k})d\underline{x}d\underline{y} = \int u(\underline{x},\underline{k}')\psi^*(\underline{x},\underline{k})d\underline{x}.$$

To complete the proof of (11) we apply the integral equation (14), using the right side in place of $u_+(\underline{x},\underline{k})$ and its complex conjugate in place of $u_+^*(\underline{x},\underline{k})$, to form

$$\int u_+^*(\underline{x},\underline{k})u_+(\underline{x},\underline{k}')d\underline{x} = \delta(\underline{k}-\underline{k}')$$

$$(20)\qquad - \frac{1}{(4\pi)(2\pi)^{3/2}} \int \frac{e^{-i\underline{k}\cdot\underline{x} + ik'|\underline{x}-\underline{x}'|}}{|\underline{x}-\underline{x}'|}\psi_+(\underline{x}',\underline{k}')d\underline{x}'d\underline{x}$$

$$- \frac{1}{(4\pi)(2\pi)^{3/2}} \int \frac{e^{i\underline{k}'\cdot\underline{x} - ik|\underline{x}-\underline{x}'|}}{|\underline{x}-\underline{x}'|}\psi_+(\underline{x}',\underline{k})d\underline{x}'d\underline{x}$$

$$+ \frac{1}{(4\pi)^2} \int \frac{e^{ik'|\underline{x}-\underline{y}'| - ik|\underline{x}-\underline{x}'|}}{|\underline{x}-\underline{y}'||\underline{x}-\underline{x}'|}\psi_+^*(\underline{x}',\underline{k})\psi_+(\underline{y}',\underline{k}')d\underline{x}'d\underline{y}'d\underline{x}.$$

We now consider the last term on the right of (20) and apply (15) to the functions which are factors of ψ_+ and ψ_+^* in its integrand. We have

$$\frac{1}{(4\pi)^2}\int \frac{e^{-ik|\underline{x}-\underline{x}'|+ik'|\underline{x}-\underline{y}'|}}{|\underline{x}-\underline{x}'||\underline{x}-\underline{y}'|}\psi_+^*(\underline{x}',\underline{k})\psi_+(\underline{y}',\underline{k}')d\underline{x}'d\underline{y}'d\underline{x}$$

$$= \lim_{\epsilon\to 0}\frac{1}{(2\pi)^6}\int\frac{e^{i\underline{p}\cdot(\underline{x}-\underline{x}')+i\underline{q}\cdot(\underline{x}-\underline{y}')}}{(k^2-p^2-i\epsilon)(k'^2-q^2+i\epsilon)}\psi_+^*(\underline{x}',\underline{k})\psi_+(\underline{y}',\underline{k}')d\underline{x}'d\underline{y}'d\underline{x}d\underline{p}d\underline{q}$$

$$= \lim_{\epsilon\to 0}\frac{1}{(2\pi)^3}\int\frac{e^{-i\underline{p}\cdot\underline{x}'-i\underline{q}\cdot\underline{y}'}}{(k^2-p^2-i\epsilon)(k'^2-q^2+i\epsilon)}\psi_+^*(\underline{x}',\underline{k})\psi_+(\underline{y}',\underline{k}')\delta(\underline{p}+\underline{q})d\underline{p}d\underline{q}d\underline{x}'d\underline{y}'$$

$$= \lim_{\epsilon\to 0}\frac{1}{(2\pi)^3}\int\frac{e^{i\underline{p}\cdot(\underline{y}'-\underline{x}')}}{(k^2-p^2-i\epsilon)(k'^2-p^2+i\epsilon)}\psi_+^*(\underline{x}',\underline{k})\psi_+(\underline{y}',\underline{k}')d\underline{p}d\underline{x}'d\underline{y}'$$

$$= -\lim_{\epsilon\to 0}\frac{1}{(2\pi)^3}\int\frac{e^{i\underline{p}\cdot(\underline{y}'-\underline{x}')}}{(p^2-k^2+i\epsilon)(k'^2-p^2+i\epsilon)}\psi_+^*(\underline{x}',\underline{k})\psi_+(\underline{y}',\underline{k}')d\underline{p}d\underline{x}'d\underline{y}'.$$

If we apply the identity (16) this expression becomes

$$\lim_{\epsilon\to 0}\frac{1}{(2\pi)^3}\int\frac{1}{k'^2-k^2+2i\epsilon}\left\{\int\frac{e^{i\underline{p}\cdot(\underline{y}'-\underline{x}')}}{k^2-p^2-i\epsilon}\psi_+^*(\underline{x}',\underline{k})d\underline{x}'d\underline{p}\right\}\psi_+(\underline{y}',\underline{k}')d\underline{y}'$$

$$-\lim_{\epsilon\to 0}\frac{1}{(2\pi)^3}\int\frac{1}{k'^2-k^2+2i\epsilon}\left\{\int\frac{e^{i\underline{p}\cdot(\underline{y}'-\underline{x}')}}{k'^2-p^2+i\epsilon}\psi_+(\underline{y}',\underline{k}')d\underline{y}'d\underline{p}\right\}\psi_+^*(\underline{x}',\underline{k})d\underline{x}'.$$

If we now use (17) and (18) it becomes

$$-\lim_{\epsilon \to 0} \frac{1}{k'^2-k^2+2i\epsilon} \int \left\{ \frac{1}{4\pi} \int \frac{e^{-ik|\underline{y}'-\underline{x}'|}}{|\underline{y}'-\underline{x}'|} \psi_+^*(\underline{x}',\underline{k})d\underline{x}' \right\} \psi_+(\underline{y}',\underline{k}')d\underline{y}'$$

$$+\lim_{\epsilon \to 0} \frac{1}{k'^2-k^2+2i\epsilon} \int \left\{ \frac{1}{4\pi} \int \frac{e^{ik'|\underline{x}'-\underline{y}'|}}{|\underline{x}'-\underline{y}'|} \psi_+(\underline{y}',\underline{k}')d\underline{y}' \right\} \psi_+^*(\underline{x}',\underline{k})d\underline{x}'.$$

We now substitute from (14) for the bracketed expressions and obtain

$$-\lim_{\epsilon \to 0} \frac{1}{k'^2-k^2+2i\epsilon} \int \frac{e^{-i\underline{k}\cdot\underline{y}'}}{(2\pi)^{3/2}} \psi_+(\underline{y}',\underline{k}')d\underline{y}'$$

$$+\lim_{\epsilon \to 0} \frac{1}{k'^2-k^2+2i\epsilon} \int u_+^*(\underline{y}',\underline{k})\psi_+(\underline{y}',\underline{k}')d\underline{y}'$$

$$+\lim_{\epsilon \to 0} \frac{1}{k'^2-k^2+2i\epsilon} \int \frac{e^{i\underline{k}'\cdot\underline{x}'}}{(2\pi)^{3/2}} \psi_+^*(\underline{x}',\underline{k})d\underline{x}'$$

$$-\lim_{\epsilon \to 0} \frac{1}{k'^2-k^2+2i\epsilon} \int u_+(\underline{x}',\underline{k}')\psi_+^*(\underline{x}',\underline{k})d\underline{x}'.$$

Now by (19) the second and fourth terms in this expression cancel each other. Hence we have finally for the last term on the right side of (20) the expression

$$-\lim_{\epsilon \to 0} \frac{1}{k'^2-k^2+2i\epsilon} \int \frac{e^{-i\underline{k}\cdot\underline{y}'}}{(2\pi)^{3/2}} \psi_+(\underline{y}',\underline{k}')d\underline{y}'$$

$$+\lim_{\epsilon \to 0} \frac{1}{k'^2-k^2+2i\epsilon} \int \frac{e^{i\underline{k}'\cdot\underline{x}'}}{(2\pi)^{3/2}} \psi_+^*(\underline{x}',\underline{k})d\underline{x}'.$$

If we make use of (17) and (18) we see that the two terms in this expression cancel the second and third terms on the right of (20), and the relation (11) which we set out to prove now follows.

We can define more general eigenfunctions of $-\Delta$ and, correspondingly, more general eigenfunctions of $-\Delta + V$ can be defined by means of integral equations similar to (10). For example, we can define the eigenfunction

$$(21) \qquad u_{o+}(\underline{x},\underline{k}) = \frac{1}{(2\pi)^{3/2}} \int e^{i\underline{k}\cdot\underline{x}}\, m_+(\underline{k}',\underline{k})d\underline{k}'$$

of $-\Delta$. The function $u_{o+}(\underline{x},\underline{k})$ will be an eigenfunction of $-\Delta$ if we assume that $m_+(\underline{k}',\underline{k})$ has the form

$$(21a) \qquad m_+(\underline{k}',\underline{k}) = \mu_+(\underline{k}',\underline{k}) \;\; \delta(k'^2 - k^2),$$

according to the remarks in (2.g).

We can then define a corresponding eigenfunction $u(\underline{x},\underline{k})$ of $-\Delta + V$ as the solution of the integral equation

$$(22) \qquad u(\underline{x},\underline{k}) = u_{o+}(\underline{x},\underline{k}) - \frac{1}{4\pi} \int \frac{e^{ik|\underline{x}-\underline{x}'|}}{|\underline{x}-\underline{x}'|}\, v(\underline{x}',\underline{x}'')u(\underline{x}'',\underline{k})d\underline{x}'d\underline{x}''.$$

If we replace $\underline{k}$ by $\underline{k}'$ in (10), multiply through by $m_+(\underline{k},\underline{k}')$ and integrate over $\underline{k}'$, we see from the uniqueness of the solution of (22) and of (10) that

$$(23) \qquad u(\underline{x},\underline{k}) = \int u_+(\underline{x},\underline{k}')m_+(\underline{k}',\underline{k})d\underline{k}'.$$

Expressed in terms of abstract operators (23) is

$$(24) \qquad U = U_+M_+,$$

wherein U is the operator corresponding to the kernel $u(\underline{x},\underline{k})$ and M_+ is the operator corresponding to the kernel $m_+(\underline{k},\underline{k}')$.

We would also like to define an operator M_- such that

$$(25) \qquad U = U_-M_-.$$

We can do this by showing that the eigenfunction $u(\underline{x},\underline{k})$ also satisfies an integral equation

$$(26) \qquad u(\underline{x},\underline{k}) = u_{o-}(\underline{x},\underline{k}) - \frac{1}{4\pi}\int \frac{e^{-ik|\underline{x}-\underline{x}'|}}{|\underline{x}-\underline{x}'|} v(\underline{x}',\underline{x}'')u(\underline{x}'',\underline{k})d\underline{x}'d\underline{x}'',$$

where $u_{o-}(\underline{x},\underline{k})$ is an eigenfunction of $-\Delta$ given by

$$(27) \qquad u_{o-}(\underline{x},\underline{k}) = \frac{1}{(2\pi)^{3/2}}\int e^{i\underline{k}'\cdot\underline{x}}\, m_-(\underline{k}',\underline{k})d\underline{k}'.$$

Here, again, $m_-(\underline{k}',\underline{k})$ must have the form

$$(27a) \qquad m_-(\underline{k}',\underline{k}) = \mu_-(\underline{k}',\underline{k})\delta(k'^2 - k^2).$$

We can obtain a relation for $m_-(\underline{k}',\underline{k})$ by setting the expression (26) for $u(\underline{x},\underline{k})$ equal to the expression (22). We have then

$$(28)\qquad u_{o-}(\underline{x},\underline{k}) = u_{o+}(\underline{x},\underline{k}) - \frac{1}{4\pi}\int \left\{\frac{e^{ik|\underline{x}-\underline{x}'|}}{|\underline{x}-\underline{x}'|} - \frac{e^{-ik|\underline{x}-\underline{x}'|}}{|\underline{x}-\underline{x}'|}\right\} \times v(\underline{x}',\underline{x}'')u(\underline{x}'',\underline{k})d\underline{x}'d\underline{x}''.$$

We now multiply (28) by $\frac{e^{-i\underline{k}'\cdot\underline{x}}}{(2\pi)^{3/2}}$ and integrate with respect to $\underline{x}$.

According to (27) and (23) the result is

$$(29)\qquad m_-(\underline{k}',\underline{k}) = m_+(\underline{k}',\underline{k}) - \frac{1}{4\pi}\int \frac{e^{-i\underline{k}'\cdot\underline{x}}}{(2\pi)^{3/2}}\left\{\frac{e^{ik|\underline{x}-\underline{x}'|}}{|\underline{x}-\underline{x}'|} - \frac{e^{-ik|\underline{x}-\underline{x}'|}}{|\underline{x}-\underline{x}'|}\right\} \times v(\underline{x}',\underline{x}'')u(\underline{x}'',\underline{k})d\underline{x}'d\underline{x}''d\underline{x}.$$

It is clear from (28) and the properties of the Green's functions $\frac{e^{\pm ik(\underline{x}-\underline{x}')}}{|\underline{x}-\underline{x}'|}$ that $u_{o-}(\underline{x},\underline{k})$ is an eigenfunction of $-\Delta$. Hence, the form (27a) of $m_-(\underline{k}',\underline{k})$ we have assumed must be correct.

We can prove this more directly from (29) by making use of (17) and (18). Thus, in place of (29) we have

$$(30)\quad m_-(\underline{k}',\underline{k}) = m_+(\underline{k}',\underline{k}) + \lim_{\epsilon\to 0}\frac{1}{(2\pi)^{3/2}}\int\left\{\frac{e^{-i\underline{k}'\cdot\underline{x}'}}{k^2-k'^2+i\epsilon} - \frac{e^{-i\underline{k}'\cdot\underline{x}'}}{k^2-k'^2-i\epsilon}\right\}\psi(\underline{x}',\underline{k})d\underline{x}'$$

$$= m_+(\underline{k}',\underline{k}) - \frac{1}{(2\pi)^{3/2}}\lim_{\epsilon\to 0} 2i\epsilon\int\frac{e^{-i\underline{k}'\cdot\underline{x}'}}{(k^2-k'^2)^2+\epsilon^2}\psi(\underline{x}',\underline{k})d\underline{x}'$$

$$= m_+(\underline{k}',\underline{k}) - 2\pi i\int\frac{e^{-i\underline{k}'\cdot\underline{x}'}}{(2\pi)^{3/2}}\psi(\underline{x}',\underline{k})d\underline{x}'\delta(k'^2-k^2),$$

where we have used the identity

$$(31) \qquad \lim_{\epsilon \to 0} \frac{\epsilon}{x^2 + \epsilon^2} = \pi\delta(x).$$

The form (27a) for $m_-(\underline{k}',\underline{k})$ follows from (30) and the corresponding form (21a) for $m_+(\underline{k}',\underline{k})$.

If either one of the operators U or M_+ is known to have an inverse then (24) implies the existence of the inverse of the other. Then from (25) the existence of the inverse of M_- follows. If in (24) U is set equal to U_+ then

$$M_+ = I.$$

In this case (30) becomes

$$(32) \quad m_-(\underline{k}',\underline{k}) = \delta(\underline{k}' - \underline{k}) - 2\pi i \int \frac{e^{-ik'\cdot x'}}{(2\pi)^{3/2}} \psi_+(\underline{x}',\underline{k})d\underline{x}'\delta(k'^2 - k^2) \equiv S.$$

The operator S defined by (32) is called the scattering operator (from the definition employed in quantum mechanics), and from what we have just observed it has the form

$$(32a) \qquad S \equiv s(k',k) = \sigma(k',k)\delta(k'^2 - k^2).$$

The forms (21a), (27a) and (32a) imply in each case, we recall (cf. 2.g), that the operator commutes with the Laplacian Δ.

The relationship between the kernel of S given in (32) and the asymptotic value of $u_+(\underline{x},\underline{k})$, which may be interpreted as the scattering amplitude for an incident plane wave, can be seen at once from a consideration of (6) in the limit of large x. We observe that

$$|\underline{x}-\underline{x}'| = \sqrt{x^2 - 2\underline{x}\cdot\underline{x}' + x'^2} \sim x - \frac{\underline{x}\cdot\underline{x}'}{x} = x - \underline{x}_o\cdot\underline{x}',$$

where $\underline{x}_o$ is a unit vector in the direction of $\underline{x}$. Then from (6),

$$(33)\qquad u_+(\underline{x},\underline{k}) \sim \frac{e^{i\underline{k}\cdot\underline{x}}}{(2\pi)^{3/2}} - \frac{(2\pi)^{3/2}}{4\pi}\int \frac{e^{-ik\underline{x}_o\cdot\underline{x}'}}{(2\pi)^{3/2}} v(\underline{x}',\underline{x}'')u_+(\underline{x}'',\underline{k})d\underline{x}'d\underline{x}''\frac{e^{ikx}}{x}.$$

The factor of $\frac{e^{ikx}}{x}$ in the second term on the right of (33) is the scattering amplitude $-\frac{(2\pi)^{3/2}}{4\pi}\, t_+(\underline{k}',\underline{k})$, where

$$(34)\qquad t_+(\underline{k}',\underline{k}) = \int \frac{e^{-i\underline{k}'\cdot\underline{x}'}}{(2\pi)^{3/2}} v(\underline{x}',\underline{x}'')u_+(\underline{x}'',\underline{k})d\underline{x}'d\underline{x}'' ,$$

and it is assumed that $k'=k$ and $\underline{k}'$ has the direction of $\underline{x}$. Then we can write for the kernel $s(\underline{k}',\underline{k})$ of S the relation

$$(35)\qquad s(\underline{k}',\underline{k}) = \delta(\underline{k}'-\underline{k}) - 2\pi i t_+(\underline{k}',\underline{k})\delta(k'^2 - k^2).$$

We have from (25), (32) and the manner of deriving the relation for S that

$$(36)\qquad U_+ = U_- S,$$

and hence that

$$S = U_-^* U_+ . \tag{37}$$

We have used the fact that U_- is unitary in deriving (37), which in turn implies that S is unitary:

$$SS^* = (U_-^*U_+)(U_-^*U_+)^* = U_-^*(U_+U_+^*)U_- = U_-^*U_- = I.$$

We can conclude from this that S is unitary because we have assumed that U_+ and U_- have left and right inverses.

From (24) and (25) we have

$$U_+M_+ = U_-M_- .$$

Thus,

$$M_- = U_-^*U_+M_+ = SM_+ ,$$

from which we get

$$S = M_-M_+^{-1} . \tag{38}$$

We can now state an integral transform theorem in terms of an arbitrary eigenfunction $u(\underline{x},\underline{k})$ of the operator $-\Delta + V$.

Given an arbitrary function $f(\underline{x})$ of the type we consider as representing the abstract linear space (e.g., square integrable functions)

we can write

$$(39) \qquad f(\underline{x}) = \int u(\underline{x},\underline{k})\varphi(\underline{k})d\underline{k},$$

where

$$(40) \qquad \varphi(\underline{k}) = \int f(\underline{x}) \left\{ \int w(\underline{k},\underline{k}')u^*(\underline{x},\underline{k}')d\underline{k}' \right\} d\underline{x}$$

and the function

$$\int w(\underline{k},\underline{k}')u(\underline{x},\underline{k}')d\underline{k}'$$

is an eigenfunction of $-\Delta + V$. This can be guaranteed by requiring that $w(\underline{k},\underline{k}')$ have the form

$$(41) \qquad w(\underline{k},\underline{k}') = \omega(\underline{k},\underline{k}')\delta(k^2 - k'^2).$$

It will also turn out that the operator W, which we shall refer to as the spectral weight operator, corresponding to $w(k,k')$, is Hermitian and positive definite.

We can define W abstractly by

$$(42) \qquad W = M_+^{-1}M_+^{*-1} \ ;$$

then from (24) and (25) we obtain

$$(43) \qquad \overline{W} = M_-^{-1}M_-^{*-1} \ .$$

The fact that W as defined by (42) or (43) is Hermitian and positive definite is obvious.

The relations (24) and (25) also imply

(44) $$WU^*U = I$$

and

(45) $$UWU^* = I.$$

Relations (44) and (45) are equivalent to the transform theorem relations (39) and (40).

From (42) or (43) we see that the spectral weight operator W needed for the integral transform theorem corresponding to a given eigenfunction $u(\underline{x},\underline{k})$ can be determined if we know the integral equation of the form (22) which $u(\underline{x},\underline{k})$ satisfies. We can also obtain the spectral weight operator W in terms of the asymptotic amplitude of $u(\underline{x},\underline{k})$ as x becomes large. The scattering operator S plays a significant role in this relationship, and we can therefore obtain a direct connection between S and W.

Consider, for example, an eigenfunction

(46) $$u(\underline{x},\underline{k}) = \int u_+(\underline{x},\underline{k}')m_+(\underline{k}',\underline{k})d\underline{k}'$$

defined to have the asymptotic behavior

$$u(\underline{x},\underline{k}) \sim e^{i\underline{k}\cdot\underline{x}}$$

as x becomes large in a "forward" direction, that is, for the polar angle

θ of $\underline{x}$ in $0 \leq \theta \leq \frac{\pi}{2}$. We can write instead of (46), with $m_+^{-1}(\underline{k}',\underline{k})$ as the kernel corresponding to M_+^{-1},

$$\text{(47)} \qquad \int u(\underline{x},\underline{k}')m_+^{-1}(\underline{k}',\underline{k})d\underline{k}' = u_+(\underline{x},\underline{k}).$$

We now consider the asymptotic form of (47) for large x, using (33) and (34) and (A.1) of Appendix II. We have, assuming that $m_+^{-1}(\underline{k}',\underline{k})$ has the form

$$m_+^{-1}(\underline{k}',\underline{k}) = \mu_+^{-1}(\underline{k}',\underline{k})\delta(k'^2 - k^2),$$

$$\text{(48)} \qquad \int u(\underline{x},\underline{k}')m_+^{-1}(\underline{k}',\underline{k})d\underline{k}' \sim \int e^{i\underline{k}'\cdot\underline{x}}m_+^{-1}(\underline{k}',\underline{k})d\underline{k}'$$

$$= \int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} \frac{e^{ik'\cdot x}}{(2\pi)^{3/2}} m_+^{-1}(\underline{k}',\underline{k})k'^2 \sin\theta' dk' d\theta' d\varphi'$$

$$\sim \frac{\pi i}{(2\pi)^{3/2}} \int_0^{2\pi} d\varphi' \int_0^{\pi} d\theta' \int_0^{\infty} dk' \mu_+^{-1}(\underline{k}',\underline{k})$$

$$\times \left\{ -\delta(k'-k)\delta(\theta'-\theta)\delta(\varphi'-\varphi)\frac{e^{ikx}}{x} + \delta(k'-k)\delta(\pi-\theta-\theta')\delta(\varphi\pm\pi-\varphi')\frac{e^{-ikx}}{x} \right\}$$

$$= \frac{\pi i}{(2\pi)^{3/2}} \left\{ -\mu_+^{-1}(\underline{k}_0,\underline{k})\frac{e^{ikx}}{x} + \mu_+^{-1}(-\underline{k}_0,\underline{k})\frac{e^{-ikx}}{x} \right\},$$

where $\underline{k}_0$ is the vector $\frac{k}{x}\,\underline{x}$. In the calculation of (48) we have used the identity

$$\delta(k'^2 - k^2) = \delta([k'+k][k'-k]) = \frac{1}{2k}\,\delta(k'-k).$$

We also have from (33)

$$(49)\quad u_+(\underline{x},\underline{k}) \sim \frac{2\pi i}{(2\pi)^{3/2}} \left\{ -\delta(\theta-\theta_o)\delta(\varphi-\varphi_o)\frac{e^{ikx}}{x} + \delta(\pi-\theta-\theta_o)\delta(\varphi\pm\pi-\varphi_o)\frac{e^{-ikx}}{x} \right\}$$

$$\frac{-(2\pi)^{3/2}}{4\pi} t_+(\underline{k}_o,\underline{k})\frac{e^{ikx}}{x} .$$

By combining (47), (48) and (49) we obtain

$$(50)\quad \begin{aligned} \mu_+^{-1}(\underline{k}_o,\underline{k}) &= \frac{2k}{k^2\sin\theta}\delta(\theta-\theta_o)\delta(\varphi-\varphi_o) - 2\pi i t_+(\underline{k}_o,\underline{k}), \\ \mu_+^{-1}(-\underline{k}_o,\underline{k}) &= \frac{2k}{k^2\sin\theta}\delta(\pi-\theta-\theta_o)\delta(\varphi\pm\pi-\varphi_o). \end{aligned}$$

Finally, from (35), (50) and the identity

$$\delta(\underline{k}'-\underline{k}) = \frac{\delta(k'-k)\delta(\theta'-\theta)\delta(\varphi'-\varphi)}{k^2\sin\theta}$$

we have

$$(51)\quad \begin{aligned} m_+^{-1}(\underline{k}_o,\underline{k}) &= \delta(\underline{k}_o-\underline{k}) - 2\pi i t_+(\underline{k}_o,\underline{k})\delta(k_o^2-k^2) = s(\underline{k}_o,\underline{k}), \quad 0\le\theta_o\le\frac{\pi}{2} \\ m_+^{-1}(-\underline{k}_o,\underline{k}) &= \delta(\underline{k}_o+\underline{k}) \quad \text{or} \quad m_+^{-1}(\underline{k}_o,\underline{k}) = \delta(\underline{k}_o-\underline{k}), \quad \frac{\pi}{2}\le\theta\le\pi. \end{aligned}$$

That is,

$$(52)\quad m_+^{-1}(\underline{k}_o,\underline{k}) = s(\underline{k}_o,\underline{k})\eta\left(\frac{\pi}{2}-\theta_o\right) + \delta(\underline{k}_o-\underline{k})\eta\left(\theta_o-\frac{\pi}{2}\right).$$

Then from (42) we obtain the kernel corresponding to W in the form

$$w(\underline{k}_o,\underline{k}) = \int m_+^{-1}(\underline{k}_o,\underline{k}')m_+^{*-1}(\underline{k},\underline{k}')d\underline{k}'$$

$$= \int s(\underline{k}_o,\underline{k}')s^*(\underline{k},\underline{k}')d\underline{k}'\eta\left(\frac{\pi}{2}-\theta_o\right)\eta\left(\frac{\pi}{2}-\theta\right) + s(\underline{k}_o,\underline{k})\eta\left(\frac{\pi}{2}-\theta_o\right)\eta\left(\theta-\frac{\pi}{2}\right)$$

(53)

$$+ s^*(\underline{k},\underline{k}_o)\eta\left(\frac{\pi}{2}-\theta\right)\eta\left(\theta_o-\frac{\pi}{2}\right) + \delta(\underline{k}_o-\underline{k})\,\eta\left(\theta_o-\frac{\pi}{2}\right)$$

$$= \delta(\underline{k}_o-\underline{k}) + s(\underline{k}_o,\underline{k})\eta\left(\frac{\pi}{2}-\theta_o\right)\eta\left(\theta-\frac{\pi}{2}\right) + s^*(\underline{k},\underline{k}_o)\eta\left(\frac{\pi}{2}-\theta\right)\eta\left(\theta_o-\frac{\pi}{2}\right),$$

wherein we have used the unitary property of the scattering operator S.

4. The inverse problem

In the inverse problem our goal is to obtain the operator V of (2), that is, the kernel $v(\underline{x},\underline{x}')$, from a knowledge of the scattering operator S, or actually from a knowledge of some part of S. It will be convenient to rephrase the problem in a different form in terms of the spectral weight operator W. We can do this by making use of (53), and we observe that $w(\underline{k}_o,\underline{k})$ corresponding to W requires only part of S for its complete specification.

In the new form the inverse problem is to find V from a knowledge of W. For this purpose it will be convenient to rewrite (44) and (45) in the form

(54) $$WU^* = U_o$$

(55) $$U_o = U^{-1}\,.$$

If a solution U of (54) and (55) can be found then V will be given by

$$V = -U \Delta U_o + \Delta . \tag{56}$$

We can see this by applying $-U \Delta U_o$ to the function

$$u(\underline{x},\underline{x}') = \int u(\underline{x},\underline{k}) \frac{e^{i\underline{k}\cdot\underline{x}'}}{(2\pi)^{3/2}} d\underline{k} \equiv U ,$$

where here $u(\underline{x},\underline{x}')$ is the kernel of the integral operator representation of U on the space of functions of $\underline{x}$. By (55) the result will be

$$-\int u(\underline{x},\underline{x}') \Delta' \frac{e^{i\underline{k}\cdot\underline{x}'}}{(2\pi)^{3/2}} d\underline{x}' = -k^2 \int u(\underline{x},\underline{x}') \frac{e^{i\underline{k}\cdot\underline{x}'}}{(2\pi)^{3/2}} d\underline{x}' = -k^2 u(\underline{x},\underline{k}).$$

That is, we shall have the expression equivalent to (56):

$$-\Delta u (\underline{x},\underline{k}) + \int v(\underline{x},\underline{x}') u(\underline{x}',\underline{k}) d\underline{x}' = k^2 u(\underline{x},\underline{k}) ,$$

where $v(\underline{x},\underline{x}')$ is the kernel corresponding to the operator V given by (56).

From (54) and (56) it follows, since Δ is Hermitian, that if V is Hermitian, i.e.,

$$V = V^* , \tag{57}$$

then

$$U \Delta W U^* = U W \Delta U^* ,$$

where we have used the fact that W is Hermitian. Since, according to (55), U has an inverse it follows that

(58) $$\Delta W = W\Delta ,$$

which means that the kernel corresponding to W in the $\underline{k}$ representation has the form

(59) $$w(\underline{k},\underline{k}')\delta(k^2 - k'^2)$$

as we have assumed. Relations (58) and (59) are, in fact, necessary and sufficient for (57) to hold.

Obviously the solution of (54), (55) for U is not unique for a given W. As we have just observed when W has the form (59), which we always assume, then any V corresponding to a solution U will automatically be Hermitian. In order to guarantee that V be truly self-adjoint and that the correct scattering operator be reproduced we must introduce other conditions. The general question of the uniqueness and existence of V under any general condition which leads to the correct scattering operator is still open, and we shall not attempt to settle it here. Instead we shall consider a particular condition which leads to a unique V such that the part of the scattering operator needed to define W, as in (53), will be reproduced.

The condition we shall prescribe is a restriction on the form of the kernel $u(\underline{x},\underline{x}')$. We write, first,

(60) $$U = I + K ,$$

and then assume that K is represented by a kernel of the form

(61) $$K \equiv q(\underline{x},\underline{x}')\eta(x'-x) .$$

Then

$$u(\underline{x},\underline{x}') = \delta(\underline{x}-\underline{x}') + q(\underline{x},\underline{x}')\eta(x'-x)$$

and

(62) $$u(\underline{x},\underline{k}) = \frac{e^{i\underline{k}\cdot\underline{x}}}{(2\pi)^{3/2}} + \int_x^\infty \int_0^{2\pi} \int_0^\pi q(\underline{x},\underline{x}') \frac{e^{ik\cdot x'}}{(2\pi)^{3/2}} x'\sin\theta' d\theta' d\varphi' dx'.$$

From (62) we have formally

(63) $$u(\underline{x},\underline{k}) \sim \frac{e^{i\underline{k}\cdot\underline{x}}}{(2\pi)^{3/2}}$$

for large x as required.

Now from (61) we see that

(64) $$K^* \equiv \tilde{q}(\underline{x},\underline{x}') = q^*(\underline{x}',\underline{x})\eta(x-x').$$

We shall prove that the assumption involved in (60), (61) and (64) leads to the following theorem:*

For any U, U_o of the form given in (60), (61) and satisfying the equation (54),

$$WU^* = U_o ,$$

*The proof of this theorem was furnished by Professors J. B. Keller and B. Friedman.

we have the relation

$$UU_o = I \; ; \tag{65}$$

i.e., if U has an inverse the equation (55)

$$U_o = U^{-1}$$

will be satisfied automatically.

<u>PROOF</u>: We have

$$UU_o = UWU^* = U_o^* U^* , \tag{66}$$

since by (42) W is Hermitian. On setting

$$U_o = I + K_o , \tag{67}$$

where

$$K_o \equiv q_o(\underline{x},\underline{x}')\eta(x' - x) , \tag{68}$$

we have from (66)

$$K + K_o + KK_o = K^* + K_o^* + K_o^* K^* . \tag{69}$$

In terms of q and q_o this is

$$q(\underline{x},\underline{x}')\eta(x'-x) + q_o(\underline{x},\underline{x}')\eta(x'-x) + \int q(\underline{x},\underline{x}'')\eta(x''-x)q_o(x'',x')\eta(x'-x'')dx''$$

$$= q^*(\underline{x}',\underline{x})\eta(x-x') + q_o^*(\underline{x}',\underline{x})\eta(x-x') + \int q_o^*(\underline{x}',\underline{x}'')\eta(x''-x')q^*(\underline{x}'',\underline{x})\eta(x-x'')dx''.$$

Each term on the left vanishes for $x' < x$, and each term on the right vanishes for $x > x'$. It follows that both sides are zero identically. Thus, we have proved that

$$UU_o = I + K + K_o + KK_o = I$$

as desired.

We now observe that if we set

$$W = \Omega + I,$$

then equation (54) becomes

$$\Omega K^* + \Omega + K^* = K_o$$

or

$$K\Omega + \Omega + K = K_o^* . \tag{70}$$

In terms of the integral operator kernels we obtain from (70), if we use the fact that K and K_o have the form given by (61),

$$(71)\quad \int_0^{2\pi}\int_0^{\pi}\int_x^{\infty} q(\underline{x},\underline{x}'')\omega(\underline{x}'',\underline{x}')x'' \sin\theta'' dx'' d\theta'' d\varphi'' + \omega(\underline{x},\underline{x}') + q(\underline{x},\underline{x}') = 0, x' > x.$$

For each fixed $\underline{x}$ equation (71) is an integral equation for $q(\underline{x},\underline{x}')$ as a function of $\underline{x}'$ in the range $x' > x$. If equation (71) has a well-behaved solution for $q(\underline{x},\underline{x}')$ in the range $x' > x$ we can form the kernel

$$u(\underline{x},\underline{x}') = \delta(\underline{x}-\underline{x}') + q(\underline{x},\underline{x}')\eta(x'-x).$$

We can prove that (71) has a unique solution by making use of the positive definite character of the spectral weight operator W. It follows then from (63) that the solution we have given reproduces the part of the scattering operator required to form the operator W.

We can show that the solution (assumed to be bounded) $q(\underline{x},\underline{x}')$ of (71) is unique by a standard argument which goes as follows. Suppose $q_1(\underline{x},\underline{x}')$ is a second solution of (71) and that q and q_1 are bounded. Then the function

$$\gamma(\underline{x},\underline{x}') = q(\underline{x},\underline{x}') - q_1(\underline{x},\underline{x}')$$

satisfies

$$\int_0^{2\pi}\int_0^{\pi}\int_x^{\infty} \gamma(\underline{x},\underline{x}'')\omega(\underline{x}'',\underline{x}')x'' \sin\theta'' dx'' d\theta'' d\varphi'' + \gamma(\underline{x},\underline{x}') = 0, x' > x.$$

Whence

$$\int \eta(x'' - x)\gamma(\underline{x},\underline{x}'')\{\omega(\underline{x}'',\underline{x}') + \delta(\underline{x}'' - \underline{x}')\}d\underline{x}'' = 0.$$

This has the operator form

$$\Gamma W = 0.$$

We multiply (operator multiplication) on the right by Γ^* and apply to an arbitrary vector φ:

$$\Gamma W \Gamma^* \varphi = 0.$$

The inner product of φ with this is

$$(\varphi, \Gamma W \Gamma^* \varphi) = (\Gamma^* \varphi, W \Gamma^* \varphi) = 0.$$

Since W is positive definite the vanishing of the quadratic form implies

$$\Gamma^* \varphi = 0.$$

Since φ is arbitrary then

$$\Gamma^* = 0,$$

and hence

$$q(\underline{x},\underline{x}') = q_1(\underline{x},\underline{x}') ,$$

as required.

Finally, we conclude this section and the article with the remark that the relation (56) for V reduces to a simple expression in the case

in which V is represented by the kernel $v(\underline{x},\underline{x}')$. This expression gives $v(\underline{x},\underline{x}')$ directly in terms of $q(\underline{x},\underline{x}')$ which is a solution of the integral equation (71). From (56) we obtain

$$V = (\Delta K - K\Delta)K_o,$$

or equivalently

$$v(\underline{x},\underline{x}') = \Delta\{q(\underline{x},\underline{x}')\eta(x'-x)\} - \Delta'\{q(\underline{x},\underline{x}')\eta(x'-x)\}$$

$$+ \int [\Delta\{q(\underline{x},\underline{x}'')\eta(x''-x)\}]\{q_o(\underline{x}'',\underline{x})\eta(x-x'')\}d\underline{x}''$$

$$- \int \Delta''[\{q(\underline{x},\underline{x}'')\eta(x''-x)\}]\{q_o(\underline{x}'',\underline{x})\eta(x-x'')\}d\underline{x}''.$$

If we express the functions and Laplacians on the right in spherical coordinates then because of the fact that the derivative of the η function is a δ function we find that $v(\underline{x},\underline{x}')$ is a sum of two terms, one of which has $\delta(x'-x)$ as a factor and the other of which has $\eta(x'-x)$ as a factor (the terms in $\delta'(x'-x)$ cancel out). Since V must be Hermitian we have

$$v^*(\underline{x}',\underline{x}) = v(\underline{x},x'),$$

and thus the coefficient of the term in $\eta(x'-x)$ must vanish. What remains is the following expression for $v(\underline{x},\underline{x}')$:

(72) $$v(\underline{x},\underline{x}') = \frac{-2\delta(x-x')}{x^2}\frac{\partial}{\partial x}\left\{x^2 q(x,\theta,\varphi;\, x,\theta',\varphi')\right\},$$

where $q(x,\theta,\varphi;\, x,\theta',\varphi')$ is obtained from $q(\underline{x},\underline{x}')$ by using spherical coördinates and setting $x' = x$. We recall that $q(\underline{x},\underline{x}')$ is the solution of the integral equation (71) whose kernel $\omega(\underline{x},\underline{x}')$ is the representer of the operator $\Omega = W - I$ (cf.(53)).

APPENDIX I

Proof of the identity

$$-\frac{1}{4\pi}\frac{e^{\pm ik|\underline{x}-\underline{x}'|}}{|\underline{x}-\underline{x}'|} = \lim_{\epsilon\to 0}\frac{1}{(2\pi)^3}\int\frac{e^{i\underline{p}\cdot(\underline{x}-\underline{x}')}}{k^2-p^2\pm i\epsilon}\,d\underline{p}\;.$$

We have

$$\frac{1}{(2\pi)^3}\int\frac{e^{i\underline{p}\cdot(\underline{x}-\underline{x}')}}{k^2-p^2\pm i\epsilon}\,d\underline{p} = \frac{1}{(2\pi)^3}\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty}\frac{e^{ip|\underline{x}-\underline{x}'|\cos\theta}}{k^2-p^2\pm i\epsilon}\,p^2\sin\theta\,dp\,d\theta\,d\varphi$$

$$= \frac{i}{(2\pi)^2|\underline{x}-\underline{x}'|}\int_0^{\infty}\frac{e^{-ip|\underline{x}-\underline{x}'|}-e^{ip|\underline{x}-\underline{x}'|}}{k^2-p^2\pm i\epsilon}\,p\,dp$$

$$= \frac{2}{(2\pi)^2|\underline{x}-\underline{x}'|}\int_0^{\infty}\frac{p\sin p|\underline{x}-\underline{x}'|}{k^2-p^2\pm i\epsilon}\,dp = \frac{1}{(2\pi)^2|\underline{x}-\underline{x}'|}\int_{-\infty}^{\infty}\frac{p\sin p|\underline{x}-\underline{x}'|}{k^2-p^2\pm i\epsilon}\,dp$$

$$= \frac{1}{2i(2\pi)^2|\underline{x}-\underline{x}'|}\left[\int_0^\infty \frac{e^{ip|x-x'|}p\,dp}{\left(\sqrt{k^2 \pm i\epsilon}-p\right)\left(\sqrt{k^2 \pm i\epsilon}+p\right)} - \int_{-\infty}^{\infty} \frac{e^{-ip|\underline{x}-\underline{x}'|}p\,dp}{\left(\sqrt{k^2 \pm i\epsilon}-p\right)\left(\sqrt{k^2 \pm i\epsilon}+p\right)}\right]$$

$$= \frac{1}{4\pi|\underline{x}-\underline{x}'|}\left[-\tfrac{1}{2}e^{\pm ik|\underline{x}-\underline{x}'|} - \tfrac{1}{2}e^{\,ik|\underline{x}-\underline{x}'|}\right] = -\frac{1}{4\pi|\underline{x}-\underline{x}'|}\,e^{\pm ik|\underline{x}-\underline{x}'|} \;.$$

APPENDIX 2

Proof of the asymptotic relation

$$\text{(A.1)}\quad \lim_{x\to\infty} e^{i\underline{k}\cdot\underline{x}}k^2\sin\theta \sim -2i\pi k\delta(\theta-\theta')\delta(\varphi-\varphi')\frac{e^{ikx}}{x}$$

$$+2i\pi k\delta(\pi-\theta-\theta')\delta(\varphi\pm\pi-\varphi')\frac{e^{-ikx}}{x} \;.$$

We begin with a proof of the relation

$$\text{(A.2)}\quad \lim_{\alpha\to\infty}\alpha^{\frac{1}{2}}\,e^{\pm i\alpha x^2} = \sqrt{\pi}\,e^{\pm i\frac{\pi}{4}}\,\delta(x).$$

Proof of (A.2):

Consider

$$\lim_{\alpha\to\infty}\alpha^{\frac{1}{2}}\int_{-\infty}^{\infty}e^{\pm i\alpha x^2}f(x)dx = \lim_{\alpha\to\infty}\alpha^{\frac{1}{2}}\int_0^\infty e^{\pm i\alpha x^2}f(x)dx + \lim_{\alpha\to\infty}\alpha^{\frac{1}{2}}\int_{-\infty}^{0}e^{\pm i\alpha x^2}f(x)dx$$

$$= \left[\frac{1}{2}\int_0^\infty \frac{e^{\pm it}}{t^{\frac{1}{2}}}dt + \frac{1}{2}\int \frac{e^{\pm it}}{t^{\frac{1}{2}}}dt\right]f(0) = \sqrt{\pi}\,e^{\pm i\frac{\pi}{4}}\,f(0),$$

where we have used the change of variable

$$x = \alpha^{-\frac{1}{2}} t^{\frac{1}{2}} .$$

The proof of (A.1) can be carried out by the method of stationary phase. We write first

$$\underline{k}\cdot\underline{x} = kx[\cos(\varphi - \varphi')\sin\theta \sin\theta' + \cos\theta \cos\theta'].$$

Stationary points where derivatives of $\underline{k}\cdot\underline{x}$ with respect to θ and φ vanish are

$$\theta_1' = \theta, \; \varphi_1' = \varphi$$

and

$$\varphi_2' = \begin{cases} \varphi + \pi, & 0 \leq \varphi \leq \pi \\ \varphi - \pi, & \pi \leq \varphi \leq 2\pi \end{cases}, \qquad \theta_2' = \pi - \theta.$$

We consider separately two power series expansions of $\underline{k}\cdot\underline{x}$ up to quadratic terms about each of those stationary points in the θ',φ' plane. The expression $e^{i\underline{k}\cdot\underline{x}}k^2\sin\theta$ can be regarded as a sum of two terms in which the phase $\underline{k}\cdot\underline{x}$ of the exponential is replaced by the power series about each of the stationary points in turn. If this is done and (A.2) is used we arrive at

$$\lim_{x\to\infty} e^{i\underline{k}\cdot\underline{x}}k^2\sin\theta = \lim_{x\to\infty} e^{ikx}\Big[\sqrt{\pi}\, e^{-i\frac{\pi}{4}}\sqrt{\frac{2}{xk\sin\theta}}\, e^{-i\frac{\pi}{4}}\sqrt{\pi}\sqrt{\frac{2}{xk\sin\theta}} \times$$

$$\times k^2\sin\theta\,\delta(\theta-\theta')\delta(\varphi-\varphi')\Big]$$

$$+ \lim_{x\to\infty} e^{-ikx}\Big[\sqrt{\pi}\, e^{i\frac{\pi}{4}}\sqrt{\frac{2}{x\sin\theta}}\sqrt{\pi}\, e^{i\frac{\pi}{4}}\sqrt{\frac{2}{xk\sin\theta}}\, k^2\sin\theta'\delta(\pi-\theta-\theta')\delta(\varphi\pm\pi-\varphi')\Big]$$

$$= -2i\pi k\delta(\theta-\theta')\delta(\varphi-\varphi')\,\frac{e^{ikx}}{x} + 2i\pi k\delta(\pi-\theta-\theta')\delta(\varphi\pm\pi-\varphi')\,\frac{e^{-ikx}}{x}\,.$$

REFERENCES

[1] I. Kay and H. Moses, The determination of the scattering potential from the spectral measure function: I. Continuous spectrum, Il Nuovo Cim., 2, pp. 917-961 (1955).

[2] I. Kay and H. Moses, The determination of the scattering potential from the spectral measure function: V. The Gelfand Levitan equation for the three-dimensional scattering problem, Il Nuovo Cim., 22, pp. 689-705 (1961).

[3] K. Rawer, The Ionosphere, Frederick Ungar, New York, p. 17 (1957).